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The Riemann problem method for solving a perturbed nonlinear Schrödinger equation describing pulse propagation in optic fibres

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Abstract. We used the Riemann problem method with a 3×3 matrix system to find the femtosecond single soliton solution for a perturbed nonlinear Schrödinger equation which describes bright ultrashort pulse propagation in properly tailored monomode optical fibres. Compared with the Gel'fand–Levitan–Marchenko approach, the major advantage of the Riemann problem method is that it provides the general single soliton solution in a simple and compact form. Unlike the standard nonlinear Schrödinger equation, here the single soliton solution exhibits periodic evolution patterns.

1. Introduction

Optical solitons in monomode optical fibres are now at the centre of an active research field due to their remarkable stability properties. As Hasegawa and Tapert [1] have shown, in the region of the anomalous group-velocity dispersion of optical fibres it is possible to propagate *bright* solitons and in the region of normal group-velocity dispersion it is possible to excite *dark* solitons. The generation of both bright [2] and dark [3] solitons in monomode optical fibres has been demonstrated in a series of elegant experiments. The technological applications of solitons as natural bits of information make them suitable for long-distance communication systems and this has justified the long-lasting interest in this area.

The propagation of optical solitons in the picosecond domain can be well described by the nonlinear Schrödinger equation (NLSE). The NLSE is one of the completely integrable nonlinear partial differential equations and its solutions can be obtained by different methods, e.g. by using the inverse-scattering transform (IST) [4–9], the Lie group theory [10], by constructing a certain completely integrable finite-dimensional dynamical system whose solutions determine the exact solutions of the NLSE [11–13], etc. Recently within the framework of IST a perturbation theory was developed to investigate the effects of various perturbations on soliton propagation down an optical fibre [14, 15].

When the pulses are shorter than 100 fs it is necessary to include in the NLSE the higher-order nonlinear and dispersive effects, and the propagation of optical pulses in monomode optical fibres is well described by the following modified NLSE:

$$\mathrm{i}\frac{\partial q}{\partial Z} + \frac{1}{2}\frac{\partial^2 q}{\partial T^2} + |q|^2 q + \mathrm{i}\epsilon\left[\alpha_1 \frac{\partial^3 q}{\partial T^3} + \alpha_2 |q|^2 \frac{\partial q}{\partial T} + \alpha_3 q \frac{\partial |q|^2}{\partial T}\right] + \mathrm{i}\Gamma q - \sigma q \frac{\partial |q|^2}{\partial T} = 0 \tag{1.1}$$

where q represents a normalized complex amplitude of the pulse envelope, Z is the normalized distance along the fibre, T is the normalized retarded time, ϵ is a small parameter, $\alpha_1, \alpha_2, \alpha_3, \Gamma$, and σ are real normalized parameters which depend on the fibre characteristics [16–19]. The last two terms in (1.1) describe the fibre loss effect and the self-induced Raman scattering effect which continuously downshift the mean frequency of femtosecond solitons. Because the soliton self-frequency shift is inverse proportional to the fourth power of the pulse width [20], the Raman scattering effect is the most important effect for femtosecond pulses.

The necessity of including the higher-order nonlinear and dispersive effects in the NLSE was realized early [21, 22]. Since then, considerable attention has been given to these effects (see e.g. reviews [19] and the references therein) and to the nonlinear effects resulting from the delayed response of the fibre nonlinearity [23–27].

When *acting individually* the third-order dispersion term has a destructive influence on the soliton [28]. However, in the absence of the terms which account for the fibre loss and the intrapulse Raman scattering effect and for an appropriate choice of the parameters α_1, α_2, α_3, the *combined action* of higher-order nonlinear and dispersive effects can make the modified NLSE completely integrable by IST. Thus the cases when $\alpha_1 : \alpha_2 : \alpha_3 = 0 : 1 : 1$ (the derivative NLSE type I), $\alpha_1 : \alpha_2 : \alpha_3 = 0 : 1 : 0$ (the derivative NLSE type II), and $\alpha_1 : \alpha_2 : \alpha_3 = 1 : 6 : 0$ (the Hirota equation) were solved in [29–31], respectively. The case $\alpha_1 : \alpha_2 : \alpha_3 = 1 : 6 : 3$ (also an integrable one) has been recently studied in [32, 33] by using the IST in the Gel'fand–Levitan–Marchenko (GLM) approach with 3×3 U–V matrix representation. IST with 3×3 U–V matrix representation was first analysed in [34, 35].

We mention that, for a medium with an arbitrary linear dispersion law represented in a polynomial form, the Hirota equation can be further generalized to a still integrable equation which includes higher-order dispersion and nonlinear terms [36].

The paper is organized as follows. In section 2 we present in detail the Riemann approach for solving (1.1) with $\alpha_1 : \alpha_2 : \alpha_3 = 1 : 6 : 3$ and $\Gamma = \sigma = 0$. This method is the most elegant and modern technique for solving nonlinear evolution equations [8, 9]. One important advantage of this technique is that the nonlinear partial differential equations which are exactly integrable may be investigated in their perturbed form by using a perturbation theory based on the Riemann problem [37]. After that, in section 3 we find the soliton solution which corresponds to the soliton number $N = 1$ (the general single soliton solution) in a very simple and compact form. Two particular cases are written explicitly: (i) the Sasa–Satsuma soliton which represents a pulse with either one or two maxima of equal heights [32], and (ii) a single soliton solution which passes through zero several times depending on its parameters and which we called a 'breather'. The general single soliton solution ($N = 1$) has an intermediate behaviour between these two particular cases. The Riemann problem method (RPM) offers an easier way to find a very simple and compact single soliton solution, in contrast to the GLM formalism which leads [33] to a very complicated formula for the *general single soliton solution. The transformation relation between the two formulae are* also given. Finally we present our conclusions.

2. The Riemann problem method

The Riemann problem, a classical problem in the theory of functions of a complex variable, plays a central role in the theory of solitons [8, 9]. The Riemann problem is stated as follows: on a complex plane λ, let be given a closed contour γ and two sets of points: $\nu_1, \ldots, \nu_n$ inside γ, and $\mu_1, \ldots, \mu_n$ outside γ. Let be $G(\lambda)$ a matrix function defined on γ, and two sets of subspaces N_i, M_i such that $N_i \oplus M_i = \mathbb{C}^N$, $i = 1, \ldots, n$. It is required to find two matrix functions $\psi_1(\lambda)$ and $\psi_2(\lambda)$, with the normalization $\psi_2(\infty) = I$ (I being the $N \times N$ unit matrix), analytical inside and outside γ, respectively, such that $\psi_1(\lambda)\psi_2(\lambda) = G(\lambda)$ on γ and Im $\psi_1(\nu_i) = N_i$, Ker $\psi_2(\mu_i) = M_i$, $i = 1, \dots, n$.

The perturbed NLSE (1.1) for $\alpha_1 = 1$, $\alpha_2 = 6$, $\alpha_3 = 3$, and $\sigma = \Gamma = 0$:

$$\mathrm{i}\frac{\partial q}{\partial Z} + \frac{1}{2}\frac{\partial^2 q}{\partial T^2} + |q|^2 q + \mathrm{i}\epsilon\left[\frac{\partial^3 q}{\partial T^3} + 6|q|^2\frac{\partial q}{\partial T} + 3q\frac{\partial |q|^2}{\partial T}\right] = 0 \tag{2.1}$$

with the transformation

$$\begin{aligned} u(x,t) &= q(T,Z)\exp\left[-\frac{\mathrm{i}}{6\epsilon}(T - \frac{Z}{18\epsilon})\right] \\ t &= Z \\ x &= T - \frac{Z}{12\epsilon} \end{aligned} \tag{2.2}$$

is changed into a complex modified Korteweg–de Vries equation:

$$\frac{\partial u}{\partial t} + \epsilon\left[\frac{\partial^3 u}{\partial x^3} + 6|u|^2\frac{\partial u}{\partial x} + 3u\frac{\partial |u|^2}{\partial x}\right] = 0\,. \tag{2.3}$$

Equation (2.3) is equivalent with the so-called zero curvature condition [8]:

$$U_t - V_x + [U, V] = 0 \tag{2.4}$$

where

$$U = \begin{pmatrix} -\mathrm{i}\lambda & 0 & u \\ 0 & -\mathrm{i}\lambda & u^* \\ -u^* & -u & \mathrm{i}\lambda \end{pmatrix} \tag{2.5}$$

and

$$\begin{aligned} V = -4\mathrm{i}\epsilon\lambda^3 &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + 4\epsilon(\lambda^2 - |u|^2)\begin{pmatrix} 0 & 0 & u \\ 0 & 0 & u^* \\ -u^* & -u & 0 \end{pmatrix} \\ &+ 2\mathrm{i}\epsilon\lambda\begin{pmatrix} |u|^2 & u^2 & u_x \\ u^{*2} & |u|^2 & u_x^* \\ u_x^* & u_x & -2|u|^2 \end{pmatrix} - \epsilon\begin{pmatrix} 0 & 0 & u_{xx} \\ 0 & 0 & u_{xx}^* \\ -u_{xx}^* & -u_{xx} & 0 \end{pmatrix} \\ &+ \epsilon(uu_x^* - u_x u^*)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.6}$$

At the beginning we shall determine the x and t dependence of the subspaces N_i and M_i which are involved in the formulation of RPM. To this aim let us introduce the matrices U_0, V_0 given by

$$U_0 = \lim_{x\to\infty} U \qquad V_0 = \lim_{x\to\infty} V \tag{2.7}$$

and the following auxiliary equations:

$$-\mathrm{i}\varphi_x = (-\mathrm{i}U_0 + \tilde{U})\varphi \tag{2.8a}$$

$$-\mathrm{i}\varphi_t = (-\mathrm{i}V_0 + \tilde{V})\varphi \tag{2.8b}$$

$$\mathrm{i}\tilde{\varphi}_x = \tilde{\varphi}(-\mathrm{i}U_0 + \tilde{U}) \tag{2.9a}$$

$$\mathrm{i}\tilde{\varphi}_t = \tilde{\varphi}(-\mathrm{i}V_0 + \tilde{V}) \tag{2.9b}$$

$$-\mathrm{i}\psi_x = (-\mathrm{i}U_0 + \tilde{U})\psi + \mathrm{i}\psi U_0 \tag{2.10a}$$

$$-\mathrm{i}\psi_t = (-\mathrm{i}V_0 + \tilde{V})\psi + \mathrm{i}\psi V_0 \tag{2.10b}$$

$$\mathrm{i}\tilde{\psi}_x = \tilde{\psi}(-\mathrm{i}U_0 + \tilde{U}) + \mathrm{i}U_0\tilde{\psi} \tag{2.11a}$$

$$\mathrm{i}\tilde{\psi}_t = \tilde{\psi}(-\mathrm{i}V_0 + \tilde{V}) + \mathrm{i}V_0\tilde{\psi} \tag{2.11b}$$

where $U = U_0 + \mathrm{i}\tilde{U}$, $V = V_0 + \mathrm{i}\tilde{V}$. Let φ be a solution of the system formed by (2.8a) and (2.8b). The compatibility condition corresponding to this system is given by (2.4). Besides this we introduce the matrix $\omega(x, t; \lambda)$ as the solution of the system

$$\omega_x = U_0\omega \qquad \omega_t = V_0\omega \tag{2.12}$$

which has the compatibility condition

$$U_{0t} - V_{0x} + [U_0, V_0] = 0\,. \tag{2.13}$$

In our case

$$\omega_{ij} = \delta_{ij}\mathrm{e}^{\mathrm{i}a_j(\lambda x + 4\lambda^3 t)} \tag{2.14}$$

with $a_1 = a_2 = -1, a_3 = 1$. Then

$$\psi = \varphi\omega^{-1} \tag{2.15}$$

and similarly

$$\tilde{\psi} = \omega\tilde{\varphi}\,. \tag{2.16}$$

The Riemann problem can be written as follows:

$$\tilde{\psi}(x, t; \lambda)\psi(x, t; \lambda) = \omega(x, t; \lambda)G(\lambda)\omega^{-1}(x, t; \lambda) \tag{2.17}$$

where $G(\lambda) = \tilde{\varphi}(x, t; \lambda)\varphi(x, t; \lambda)$ and $\tilde{\varphi}^{-1}(x, t; \lambda)$ satisfy the system formed by (2.8a) and (2.8b). To make possible the derivation with respect to x and t in (2.17) in $\lambda = \infty$, where $U = V = \infty$, we have to impose $G(\infty) = I$. With the notations: $\psi_1 = \tilde{\psi}$, $\psi_2 = \psi$, $\varphi_1^{-1} = \tilde{\varphi}$, $\varphi_2 = \varphi$, equation (2.17) becomes

$$\psi_1(x, t; \lambda)\psi_2(x, t; \lambda) = \omega(x, t; \lambda)\varphi_1^{-1}(x, t; \lambda)\varphi_2(x, t; \lambda)\omega^{-1}(x, t; \lambda) \tag{2.18}$$

and $\varphi_1(x, t; \infty) = \varphi_2(x, t; \infty)$. After differentiating with respect to x and t in (2.18) we have

$$\psi_{1x}\psi_2 + \psi_1\psi_{2x} = U_0\psi_1\psi_2 - \psi_1\psi_2U_0 \tag{2.19a}$$

$$\psi_{1t}\psi_2 + \psi_1\psi_{2t} = V_0\psi_1\psi_2 - \psi_1\psi_2V_0\,. \tag{2.19b}$$

Thus we can define two matrices $\mathcal{U}$ and $\mathcal{V}$ as follows:

$$\mathcal{U} \equiv -\psi_1^{-1}(\psi_{1x} - U_0\psi_1) = (\psi_{2x} + \psi_2U_0)\psi_2^{-1} \tag{2.20a}$$

$$\mathcal{V} \equiv -\psi_1^{-1}(\psi_{1t} - V_0\psi_1) = (\psi_{2t} + \psi_2V_0)\psi_2^{-1} \tag{2.20b}$$

which have the same poles as U_0, V_0, respectively. Moreover, it is easy to demonstrate that $\mathcal{U} = U_0$ and $\mathcal{V} = V_0$. In order to find the soliton solution of (2.3) we have to consider the Riemann problem with zeros. Let ν_i, μ_i, $i = 1, \ldots, n$ be these zeros (independent on x and t), i.e. $\det\psi_1(\nu_i) = \det\psi_2(\mu_i) = 0$ and

$$N_i = \mathrm{Im}\ \psi_1|_{\lambda=\nu_i} \tag{2.21a}$$

$$M_i = \mathrm{Ker}\ \psi_2|_{\lambda=\mu_i}\,. \tag{2.21b}$$

For $\lambda = \nu_i$ or $\lambda = \mu_i$ a condition which has to be fulfilled is that the poles that can appear at these points have to be suppressed. With the notation

$$D_x^{(i)} = \partial_x - U_0|_{\lambda=\nu_i} \tag{2.22a}$$

$$D_t^{(i)} = \partial_t - V_0|_{\lambda=\nu_i} \tag{2.22b}$$

$$\tilde{D}_x^{(i)} = \partial_x - U_0|_{\lambda=\mu_i} \tag{2.22c}$$

$$\tilde{D}_t^{(i)} = \partial_t - V_0|_{\lambda=\mu_i} \tag{2.22d}$$

from the conditions

$$\lim_{\lambda\to\nu_i} (\lambda - \nu_i)U_0 = \lim_{\lambda\to\nu_i} (\lambda - \nu_i)V_0 = \lim_{\lambda\to\mu_i} (\lambda - \mu_i)U_0 = \lim_{\lambda\to\mu_i} (\lambda - \mu_i)V_0 = 0 \tag{2.23}$$

we have

$$D_{x,t}^{(i)} N_i(x,t) = 0 \tag{2.24a}$$

$$\tilde{D}_{x,t}^{(i)} M_i(x,t) = 0 \tag{2.24b}$$

which are equivalent to

$$N_i(x,t) = \omega(x,t;\nu_i)N_i^{(0)} \tag{2.25a}$$

$$M_i(x,t) = \omega(x,t;\mu_i)M_i^{(0)} \tag{2.25b}$$

where $N_i^{(0)}$ and $M_i^{(0)}$ are independent of x and t.

Having derived the time dependence (2.25) of the Riemann problem we shall turn our attention to the solving of the Riemann problem for $t = 0$. We shall write (2.8a) as

$$-\mathrm{i}\varphi_x = (J\lambda + \tilde{U})\varphi \tag{2.26}$$

with $J = \mathrm{diag}[a_1, a_2, a_3]$; $a_1 = a_2 = -1$, $a_3 = 1$. We notice that $\tilde{U}^\dagger = \tilde{U}$.

Let $(\varphi^\pm(x;\lambda))_{ik}$ be those solutions of (2.26) which satisfy the conditions

$$(\varphi^\pm(x;\lambda))_{ik} \to \delta_{ik}\mathrm{e}^{\mathrm{i}a_k\lambda x} \qquad x \to \pm\infty . \tag{2.27}$$

We introduce a unimodular transition matrix $S(\lambda)$,

$$\varphi^-(x;\lambda) = \varphi^+(x;\lambda)S(\lambda) . \tag{2.28}$$

By denoting $(S)_{ij} \equiv \alpha_{ij}$ and $(S^{-1})_{ij} \equiv \beta_{ij}$ one can write the following relations:

$$S^+ = SS^- \tag{2.29}$$

$$R^- = SR^+ \tag{2.30}$$

where

$$S^+ = \begin{pmatrix} 1 & -\beta_{12} & \alpha_{13} \\ 0 & \beta_{11} & \alpha_{23} \\ 0 & 0 & \alpha_{33} \end{pmatrix} \qquad S^- = \begin{pmatrix} \beta_{11} & 0 & 0 \\ \beta_{21} & \alpha_{33} & 0 \\ \beta_{31} & -\alpha_{32} & 1 \end{pmatrix}$$
$$R^- = \begin{pmatrix} \alpha_{11} & 0 & 0 \\ \alpha_{21} & \beta_{33} & 0 \\ \alpha_{31} & -\beta_{32} & 1 \end{pmatrix} \qquad R^+ = \begin{pmatrix} 1 & -\alpha_{12} & \beta_{13} \\ 0 & \alpha_{11} & \beta_{23} \\ 0 & 0 & \beta_{33} \end{pmatrix} . \tag{2.31}$$

We mention that the factorizations (2.29) and (2.30) are not unique, but the final results are not influenced by our specific choice (2.31).

If χ^+ and χ^- are given by

$$\chi^+ = \varphi^+ S^+ = \varphi^- S^- \tag{2.32a}$$

$$\chi^- = \varphi^- R^+ = \varphi^+ R^- \tag{2.32b}$$

then

$$\psi^+ = \chi^+ e^{-iJ\lambda x} \tag{2.33a}$$

$$\psi^- = \chi^- e^{-iJ\lambda x} \tag{2.33b}$$

are solutions of the system (2.10).

One can demonstrate that ψ^+ can be analytically continued in the lower complex half-plane λ , ψ^- can be analytically continued in the upper complex half-plane λ and:

$$\tilde{U} = \lim_{\lambda\to\infty} \lambda[\psi^\pm, J] \tag{2.34}$$

where

$$\tilde{U} = -i \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & u^* \\ -u^* & -u & 0 \end{pmatrix}.$$

This equation gives us the solution of (2.3) if we know either the matrix function ψ^+ or ψ^-.

Now we consider the equation

$$i\tilde{\varphi}_x = \tilde{\varphi}(J\lambda + \tilde{U}). \tag{2.35}$$

In the same manner as above, we can introduce the corresponding Jost functions of (2.35):

$$(\tilde{\varphi}^\pm(x;\lambda))_{ik} \to \delta_{ik} e^{-ia_k\lambda x} \qquad x \to \pm\infty. \tag{2.36}$$

Due to (2.36) and (2.27) the unimodular matrix $\tilde{S}$ which can be introduced as

$$\tilde{\varphi}^- = \tilde{S}\tilde{\varphi}^+ \tag{2.37}$$

is equal to S^{-1}. Following the same procedures as before one can write

$$\tilde{S}^+ = \tilde{S}^-\tilde{S} \tag{2.38a}$$

$$\tilde{R}^- = \tilde{R}^+\tilde{S} \tag{2.38b}$$

where $\tilde{S}^\pm$, $\tilde{R}^\pm$ are given by

$$\tilde{S}^+ = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ 0 & \alpha_{33} & -\alpha_{23} \\ 0 & 0 & 1 \end{pmatrix} \qquad \tilde{S}^- = \begin{pmatrix} 1 & 0 & 0 \\ -\beta_{21} & \beta_{11} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$
$$\tilde{R}^- = \begin{pmatrix} 1 & 0 & 0 \\ -\alpha_{21} & \alpha_{11} & 0 \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} \qquad \tilde{R}^+ = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & \beta_{33} & -\beta_{23} \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.39}$$

As before we introduce the functions $\tilde{\chi}^\pm$ given by

$$\tilde{\chi}^+ = \tilde{S}^+\tilde{\varphi}^+ = \tilde{S}^-\tilde{\varphi}^- \tag{2.40a}$$

$$\tilde{\chi}^- = \tilde{R}^+\tilde{\varphi}^- = \tilde{R}^-\tilde{\varphi}^+. \tag{2.40b}$$

Then the functions $\tilde{\psi}^\pm$ which are related to the functions $\tilde{\chi}^\pm$ by the formulae

$$\tilde{\psi}^+ = e^{iJ\lambda x}\tilde{\chi}^+ \tag{2.41a}$$

$$\tilde{\psi}^- = e^{iJ\lambda x}\tilde{\chi}^- \tag{2.41b}$$

are solutions of the system (2.11).

It can be easily shown that $\tilde{\psi}^+$ can be analytically continued in the lower complex half-plane λ and $\tilde{\psi}^-$ can be analytically continued in the upper complex half-plane λ.

According to (2.17), we have

$$\tilde{\psi}^-\psi^+ = \mathrm{e}^{\mathrm{i}J\lambda x} G_1(\lambda)\mathrm{e}^{-\mathrm{i}J\lambda x} \tag{2.42a}$$

$$\tilde{\psi}^+\psi^- = \mathrm{e}^{\mathrm{i}J\lambda x} G_2(\lambda)\mathrm{e}^{-\mathrm{i}J\lambda x} \tag{2.42b}$$

where

$$G_1(\lambda) = \tilde{R}^+(\lambda)S^-(\lambda) = \tilde{R}^-(\lambda)S^+(\lambda) \tag{2.43a}$$

$$G_2(\lambda) = \tilde{S}^-(\lambda)R^+(\lambda) = \tilde{S}^+(\lambda)R^-(\lambda)\,. \tag{2.43b}$$

The Riemann problems (2.42a) and (2.42b) are equivalent, and before starting the solving of one of these two Riemann problems, we have to consider the symmetries that occur. So, if $\varphi(x,\lambda)$ is a solution of (2.8a) then $K\varphi^*(x,-\lambda^*)K$ is also a solution of (2.8a), where

$$K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.44}$$

Thus if $\det\tilde{\psi}^+(\lambda_0) = 0$ then also $\det\tilde{\psi}^+(-\lambda_0^*) = 0$. Moreover, $U^\dagger = U$ and if $\lambda \in \mathbb{R}$ then one has

$$\tilde{S} = S^{-1} = S^\dagger \tag{2.45a}$$

$$\tilde{S}^+ = (R^-)^\dagger \tag{2.45b}$$

$$\tilde{S}^- = (R^+)^\dagger \tag{2.45c}$$

$$\tilde{R}^+ = (S^-)^\dagger \tag{2.45d}$$

$$\tilde{R}^- = (S^+)^\dagger\,. \tag{2.45e}$$

Thus $\psi^-(\lambda)$ has a zero in λ_0^* if and only if $\tilde{\psi}^+(\lambda)$ has a zero in λ_0.

In order to find the single soliton solution of (2.3) we state the Riemann problem as follows.

Let γ be the real axis of the complex plane λ, and let λ_0 be a complex number with $\mathrm{Im}'\lambda_0 > 0$. Let M_0, $\tilde{M}_0$, N_0, $\tilde{N}_0$ be four complex vectorial subspaces independent on x and t such that $M_0 \oplus N_0 = \tilde{M}_0 \oplus \tilde{N}_0 = \mathbb{C}^3$ and let $M, \tilde{M}, N, \tilde{N}$ be given by

$$M \equiv \omega(\lambda_0)M_0 = \mathrm{Ker}\psi^-(\lambda_0) \tag{2.46a}$$

$$\tilde{M} \equiv \omega(-\lambda_0^*)\,\tilde{M}_0 = \mathrm{Ker}\psi^-(-\lambda_0^*) \tag{2.46b}$$

$$N \equiv \omega(\lambda_0^*)N_0 = \mathrm{Im}\tilde{\psi}^+(\lambda_0^*) \tag{2.46c}$$

$$\tilde{N} \equiv \omega(-\lambda_0)\tilde{N}_0 = \mathrm{Im}\tilde{\psi}^+(-\lambda_0)\,. \tag{2.46d}$$

It is required to find two functions $\tilde{\psi}^+(\lambda)$ and $\psi^-(\lambda)$, analytical in the lower complex half-plane and in the upper complex half-plane, respectively, such that $\tilde{\psi}^+(\lambda)\psi^-(\lambda) = I$ on the contour γ.

The hermiticity property $U^\dagger = U$ gives

$$\tilde{\psi}^+(\lambda_0) = \left[\psi^-(\lambda_0^*)\right]^\dagger \tag{2.47}$$

which implies that $M \perp N$. By using the symmetries mentioned above we have

$$\tilde{N} = KN^* \tag{2.48a}$$

$$\tilde{N}_0 = KN_0^* \tag{2.48b}$$

$$\tilde{M} = KM^* \tag{2.48c}$$

$$\tilde{M}_0 = KM_0^*\,. \tag{2.48d}$$

In order to find $\tilde{\psi}^+(\lambda)$ and $\psi^-(\lambda)$ one can write

$$\tilde{\psi}^+(\lambda) = 1 + \frac{A_+}{\lambda - \lambda_0} + \frac{A_-}{\lambda + \lambda_0^*} = \psi^-(\lambda)^{-1} \qquad (2.49)$$

$$\psi^-(\lambda) = B_+^0 + B_+^1(\lambda - \lambda_0) + \cdots = B_-^0 + B_-^1(\lambda + \lambda_0^*) + \cdots . \qquad (2.50)$$

With

$$\operatorname{Im} A_+ = \operatorname{Ker} \psi^-(\lambda_0) = M \qquad (2.51a)$$

$$\operatorname{Im} A_- = \operatorname{Ker} \psi^-(-\lambda_0^*) = \tilde{M} \qquad (2.51b)$$

we have two alternatives of splitting $\mathbb{C}^3$. The first choice corresponds to dim $M = 1$, and the second choice corresponds to dim $M = 2$. In the next section we shall choose dim $M = 1$ due to easiness of calculations. It is easy to observe that if one chooses u to be real in (2.3) then one finds the modified Korteweg–de Vries (mKdV) equation which has well known soliton solutions. The above scheme works for it, but we have a supplementary symmetry condition, i.e. $N = KN, M = KM$, and the single soliton solution coincides with the solution which may be obtained if one choose a 2×2 U–V matrix pair.

3. Single soliton solution

In the case when dim M = dim $M_0 = 1$, M_0 and N_0 can be written as:

$$M_0 = \left\{ \mu \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} | \mu \in \mathbb{C} \right\} \qquad (3.1a)$$

$$N_0 = \left\{ \mu_1 \begin{pmatrix} 1 \\ 0 \\ -a^* \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \\ -b^* \end{pmatrix} | \mu_1, \mu_2 \in \mathbb{C} \right\} \qquad (3.1b)$$

with a, b arbitrary complex numbers.

Let $\{m_+\}$ be the basis of M, $\{m_-\}$ be the basis of $\tilde{M}$, $\{n_+^1, n_+^2\}$ be the basis of N, and $\{n_-^1, n_-^2\}$ be the basis of $\tilde{N}$. We have

$$m_+ = \begin{pmatrix} a e^{-i(\lambda_0 x + 4\lambda_0^3 t)} \\ b e^{-i(\lambda_0 x + 4\lambda_0^3 t)} \\ e^{i(\lambda_0 x + 4\lambda_0^3 t)} \end{pmatrix} \qquad (3.2a)$$

$$m_- = \begin{pmatrix} b^* e^{i(\lambda_0^* x + 4\lambda_0^{*3} t)} \\ a^* e^{i(\lambda_0^* x + 4\lambda_0^{*3} t)} \\ e^{-i(\lambda_0^* x + 4\lambda_0^{*3} t)} \end{pmatrix} \qquad (3.2b)$$

$$n_+^1 = \begin{pmatrix} e^{-i(\lambda_0^* x + 4\lambda_0^{*3} t)} \\ 0 \\ -a^* e^{i(\lambda_0^* x + 4\lambda_0^{*3} t)} \end{pmatrix} \qquad n_+^2 = \begin{pmatrix} 0 \\ e^{-i(\lambda_0^* x + 4\lambda_0^{*3} t)} \\ -b^* e^{i(\lambda_0^* x + 4\lambda_0^{*3} t)} \end{pmatrix} \qquad (3.3a)$$

$$n_-^1 = \begin{pmatrix} 0 \\ e^{i(\lambda_0 x + 4\lambda_0^3 t)} \\ -a e^{-i(\lambda_0 x + 4\lambda_0^3 t)} \end{pmatrix} \qquad n_-^2 = \begin{pmatrix} e^{i(\lambda_0 x + 4\lambda_0^3 t)} \\ 0 \\ -b e^{-i(\lambda_0 x + 4\lambda_0^3 t)} \end{pmatrix} . \qquad (3.3b)$$

Because $\operatorname{Im} A_+ = M$ and $\operatorname{Im} A_- = \tilde{M}$, one can write

$$A_\pm = m_\pm \otimes x_\pm \qquad (3.4)$$

where $x_\pm$ are certain unknown vectors. Because $M \perp N$, $\tilde{n}_\pm = m_\pm$ complete the basis $\{n^1_\pm, n^2_\pm\}$ to a basis of $\mathbb{C}^3$. From the conditions $N = \operatorname{Im} \tilde{\psi}^+(\lambda_0^*)$ and $\tilde{N} = \operatorname{Im} \tilde{\psi}^+(-\lambda_0)$ one has

$$\tilde{\psi}^+(\lambda_0^*) = n^1_+ \otimes y^1_+ + n^2_+ \otimes y^2_+ \tag{3.5a}$$

$$\tilde{\psi}^+(-\lambda_0) = n^1_- \otimes y^1_- + n^2_- \otimes y^2_- \tag{3.5b}$$

with $y^1_\pm$, $y^2_\pm$ unknown vectors. Hence we obtain the relations

$$\sum_{\alpha=1}^{3} (\tilde{n}^*_+)_\alpha (\,\tilde{\psi}^+(\lambda_0^*))_{\alpha\beta} = 0 \qquad \beta = 1, \dots, 3 \tag{3.6a}$$

$$\sum_{\alpha=1}^{3} (\tilde{n}^*_-)_\alpha (\tilde{\psi}^+(-\lambda_0))_{\alpha\beta} = 0 \qquad \beta = 1, \dots, 3\,. \tag{3.6b}$$

Substituting equations (3.6) in (2.49) we have

$$\tilde{n}^*_+ + \frac{(\tilde{n}_+, m_+)}{\lambda_0^* - \lambda_0} x_+ + \frac{(\tilde{n}_+, m_-)}{2\lambda_0^*} x_- = 0 \tag{3.7a}$$

$$\tilde{n}^*_- - \frac{(\tilde{n}_-, m_+)}{2\lambda_0} x_+ + \frac{(\tilde{n}_-, m_-)}{\lambda_0^* - \lambda_0} x_- = 0 \tag{3.7b}$$

where $(n, m) = \sum_{i=1}^{3} n_i^* m_i$. From $\tilde{\psi}^+(\lambda_0^*) = K \tilde{\psi}^{+*}(-\lambda_0) K$ it follows that $A_\pm = -K A^*_\mp K$ and $x_+ = -K x^*_-$. Substituting these in (3.7) we have

$$m^*_+ + \frac{(m_+, m_+)}{\lambda_0^* - \lambda_0} x_+ - \frac{(m_+, K m^*_+)}{2\lambda_0^*} K x^*_+ = 0 \tag{3.8a}$$

$$m^*_- + \frac{(m_-, m_-)}{\lambda_0^* - \lambda_0} x_- + \frac{(m_-, K m^*_-)}{2\lambda_0} K x^*_- = 0\,. \tag{3.8b}$$

One can observe that (3.8b) is equivalent with (3.8a). From (2.34) and by using the hermiticity property $\tilde{U}^\dagger = \tilde{U}$ we have

$$\tilde{U} = \lim_{\lambda \to \infty} \lambda [J, \tilde{\psi}^+]\,. \tag{3.9}$$

Finally, $u(x, t)$ can be written as

$$u(x, t) = -2\mathrm{i}[(m_+)_1 (x_+)_3 - (m_+)^*_2 (x_+)^*_3] = -2\mathrm{i}[(m_+)_3 (x_+)_2 - (m_+)^*_3 (x_+)^*_1]\,. \tag{3.10}$$

From (3.8a), with notations: $(m_+, m_+) = \Omega_0$ and $(m_+, K m^*_+) = \Omega$, $\Omega = \Omega_1 + \mathrm{i}\Omega_2$, $\Omega_0, \Omega_1, \Omega_2 \in \mathbb{R}$, one can derive two systems of equations:

$$\frac{\Omega_0}{\lambda_0^* - \lambda_0} (x_+)_1 - \frac{\Omega}{2\lambda_0^*} (x_+)^*_2 = -(m_+)^*_1 \qquad \frac{\Omega^*}{2\lambda_0} (x_+)_1 + \frac{\Omega_0}{\lambda_0^* - \lambda_0} (x_+)^*_2 = (m_+)_2 \tag{3.11}$$

and

$$\frac{\Omega_0}{\lambda_0^* - \lambda_0} (x_+)_3 - \frac{\Omega}{2\lambda_0^*} (x_+)^*_3 = -(m_+)^*_3 \qquad \frac{\Omega^*}{2\lambda_0} (x_+)_3 + \frac{\Omega_0}{\lambda_0^* - \lambda_0} (x_+)^*_3 = (m_+)_3\,. \tag{3.12}$$

We note that the discriminants of these systems are the same

$$\Delta = \frac{\Omega_0^2}{(\lambda_0^* - \lambda_0)^2} + \frac{|\Omega|^2}{4|\lambda_0|^2} \in \mathbb{R}\,. \tag{3.13}$$

Thus the solution $u(x,t)$ is

$$u(x,t) = \frac{2\mathrm{i}}{\Delta}\left[\frac{\Omega_0}{\lambda_0^* - \lambda_0}\left((m_+)_3(m_+)_2^* + (m_+)_3^*(m_+)_1\right) + \frac{\Omega^*}{2\lambda_0}(m_+)_2^*(m_+)_3^* - \frac{\Omega}{2\lambda_0^*}(m_+)_1(m_+)_3\right]. \tag{3.14}$$

If we write $\lambda_0 = (-\xi + \mathrm{i}\eta)/2$ with $\xi, \eta \in \mathbb{R}$, $\eta > 0$ and if we use the notations: $A_0 = \eta[x - \epsilon(\eta^2 - 3\xi^2)t]$; $B_0 = \xi[x + \epsilon(\xi^2 - 3\eta^2)t]$, then $u(x,t)$ is

$$u(x,t) = \frac{2\mathrm{i}}{\Delta}\left[\frac{\Omega_0}{-\mathrm{i}\eta}\left(b^*\mathrm{e}^{-\mathrm{i}B_0} + a\mathrm{e}^{\mathrm{i}B_0}\right) + \frac{\Omega^*}{2\lambda_0}b^* - \frac{\Omega}{2\lambda_0^*}a\right] \tag{3.15}$$

with

$$\Omega_0 = (|a|^2 + |b|^2)\mathrm{e}^{A_0} + \mathrm{e}^{-A_0} \tag{3.16a}$$

$$\Omega = 2a^*b^*\mathrm{e}^{A_0 - \mathrm{i}B_0} + \mathrm{e}^{-A_0 + \mathrm{i}B_0}. \tag{3.16b}$$

Two particular cases are extremely important. First we discuss the case $b = 0$. For this choice equation (3.15) becomes

$$u(x,t) = 2\eta\mathrm{e}^{\mathrm{i}B}\frac{\mathrm{e}^{A} + c\mathrm{e}^{-A}}{\frac{1}{|c|}\mathrm{e}^{2A} + |c|\mathrm{e}^{-2A} + 2|c|} \tag{3.17}$$

where $B = B_0 + \varphi$; $A = A_0 + \rho + \gamma$; $a = \mathrm{e}^{\rho + \mathrm{i}\varphi}$; $|c| = \mathrm{e}^{\gamma}$; $c = 1 - \mathrm{i}\eta/\xi$. This solution was obtained in [32] and represents either a pulse with one maximum ($1 < |c| < 2$) or a pulse with two maxima with equal heights ($|c| > 2$).

The second important case is obtained when one chooses $a = b$ in (3.15). In this case, $u(x,t)$ takes the form

$$u(x,t) = \sqrt{2}\xi\eta\frac{\xi\cosh(A_0+\rho)\cos(B_0+\varphi) - \eta\sinh(A_0+\rho)\sin(B_0+\varphi)}{\xi^2\cosh^2(A_0+\rho) + \eta^2\sin^2(B_0+\varphi)} \tag{3.18}$$

where $e^{\rho} = \sqrt{2}|a|$ and $\varphi = \arg a$.

We called this solution a 'breather' because $\int_{-\infty}^{\infty} u(x,t)\,\mathrm{d}x = 0$. We mention that this solution does not represent a bound state of single solitons ($N = 1$).

The general single soliton solution of (2.1) can be written in the most compact form as

$$q(T,Z) = \mathrm{e}^{\mathrm{i}[\theta + \phi + (1/6\epsilon(T - Z/18\epsilon))]}\sqrt{2}\xi\eta\frac{\xi\cosh A\cos B - \eta\sinh A\sin B}{\xi^2\mathrm{e}^{\rho-\gamma}|\cosh A|^2 + \eta^2\mathrm{e}^{\gamma-\rho}|\sin B|^2} \tag{3.19}$$

where

$$A = A_0 + \mathrm{i}\theta \qquad B = B_0 - \mathrm{i}\nu$$

$$A_0 = \eta\left[T - \epsilon\left(\eta^2 - 3\xi^2 + \frac{1}{12\epsilon^2}\right)Z\right] + \rho \qquad B_0 = \xi\left[T + \epsilon\left(\xi^2 - 3\eta^2 - \frac{1}{12\epsilon^2}\right)Z\right] + \varphi$$

$$a = \mathrm{e}^{\gamma_a + \mathrm{i}\varphi_a} \qquad b = \mathrm{e}^{\gamma_b + \mathrm{i}\varphi_b}$$

$$\nu = \frac{\gamma_a - \gamma_b}{2} \qquad \phi = \frac{\varphi_a - \varphi_b}{2}$$

$$\varphi = \frac{\varphi_a + \varphi_b}{2} \qquad \gamma = \frac{\gamma_a + \gamma_b}{2} + \ln\sqrt{2}$$

$$\theta = -\tfrac{1}{2}\arctan\left(\frac{\eta}{\xi}\tanh 2\nu\right) \qquad \mathrm{e}^{2\rho} = \mathrm{e}^{2\gamma}\sqrt{\frac{\eta^2\sinh^2 2\nu + \xi^2\cosh^2 2\nu}{\xi^2}}.$$

In [33], by using the Gel'fand–Levitan–Marchenko approach the general, multiple 'humped', single soliton solution of (2.3) was found in the form

$$
\begin{aligned}
u(x,t) = &-\frac{2\mathrm{e}^{\mathrm{i}(\varphi_a+\varphi_b)/2}}{\Delta}\mathrm{e}^{-A} \\
&\times\left\{|a_0|\left[\frac{|a_0b_0|\mathrm{e}^{-2(A+\mathrm{i}B)}}{2\lambda_0^2}-\frac{(|a_0|^2+|b_0|^2)\mathrm{e}^{-2A}}{\eta^2}-1\right]\mathrm{e}^{\mathrm{i}B}\right. \\
&+|b_0|\left[\frac{|a_0b_0|\mathrm{e}^{-2(A-\mathrm{i}B)}}{\lambda_0^*}-\frac{(|a_0|^2+|b_0|^2)\mathrm{e}^{-2A}}{2\lambda_0}\right]\frac{\mathrm{e}^{-\mathrm{i}B}}{\mathrm{i}\eta} \\
&+|b_0|\left[\frac{|a_0b_0|e^{-2(A-\mathrm{i}B)}}{2\lambda_0^{*2}}-\frac{(|a_0|^2+|b_0|^2)\mathrm{e}^{-2A}}{\eta^2}-1\right]\mathrm{e}^{-\mathrm{i}B} \\
&\left.-|a_0|\left[\frac{|a_0b_0|e^{-2(A+\mathrm{i}B)}}{\lambda_0}-\frac{(|a_0|^2+|b_0|^2)\mathrm{e}^{-2A}}{2\lambda_0^*}\right]\frac{\mathrm{e}^{\mathrm{i}B}}{\mathrm{i}\eta}\right\}
\end{aligned}
\tag{3.20}
$$

where

$$
\begin{aligned}
\Delta = &\left|\left[\frac{|a_0b_0|\mathrm{e}^{-2(A-\mathrm{i}B)}}{2\lambda_0^{*2}}-\frac{(|a_0|^2+|b_0|^2)\mathrm{e}^{-2A}}{\eta^2}-1\right]\right|^2 \\
&-\frac{1}{\eta^2}\left|\left[\frac{|a_0b_0|\mathrm{e}^{-2(A-\mathrm{i}B)}}{\lambda_0^*}-\frac{(|a_0|^2+|b_0|^2)\mathrm{e}^{-2A}}{2\lambda_0}\right]\right|^2 .
\end{aligned}
$$

Here $A=\eta[x-\epsilon(\eta^2-3\xi^2)t]$, $B=\xi[x+\epsilon(\xi^2-3\eta^2)t]+(\varphi_a-\varphi_b)/2$; $\varphi_a=\arg a_0$, and $\varphi_b=\arg b_0$.

After lengthy but simple calculations, the transformation relations between a, b in (3.15) and a_0, b_0 in (3.20) are

$$
a_0=-\frac{2\eta\lambda_0^*a}{\xi(|a|^2+|b|^2)+\mathrm{i}\eta(|a|^2-|b|^2)}\qquad b_0=-\frac{2\eta\lambda_0b^*}{\xi(|a|^2+|b|^2)+\mathrm{i}\eta(|a|^2-|b|^2)}\,. \tag{3.21}
$$

In figures 1 and 2 we show the evolution of the intensity profile $|q|^2$ for the 'breather' soliton (3.18) and a general single soliton (3.19), respectively. From (3.19), one can observe that the single soliton solution exhibits a periodic evolution pattern, unlike the standard NLSE where periodic evolution appears only for higher-order solitons ($N > 1$).

4. Conclusions

By using the RPM we have obtained the general single soliton solution for the 1:6:3 perturbed NLSE (2.1) in a most simple and compact form. Very rich in qualitatively different behaviours, this solution could become a milestone for new phenomena in the femtosecond range. The key factor in solving (2.1) is the transformation (2.2) which changes the perturbed NLSE to a complex mKdV equation (2.3). Apart from being a simple mathematical trick, the transformation (2.2) also has deep physical consequences. Unlike the standard NLSE, where the soliton is a balance between pulse chirping and broadening due to the group-velocity dispersion effect and the action of the self-phase modulation due to the nonlinear refractive index, the single soliton solution of (2.3) is a balance between pulse asymmetries caused by less destructive effects described by the third-order dispersion and the self-steepening terms. Consequently we expect that the single soliton (3.19) of the perturbed NLSE (2.1) could be observed experimentally in properly tailored optic fibres.

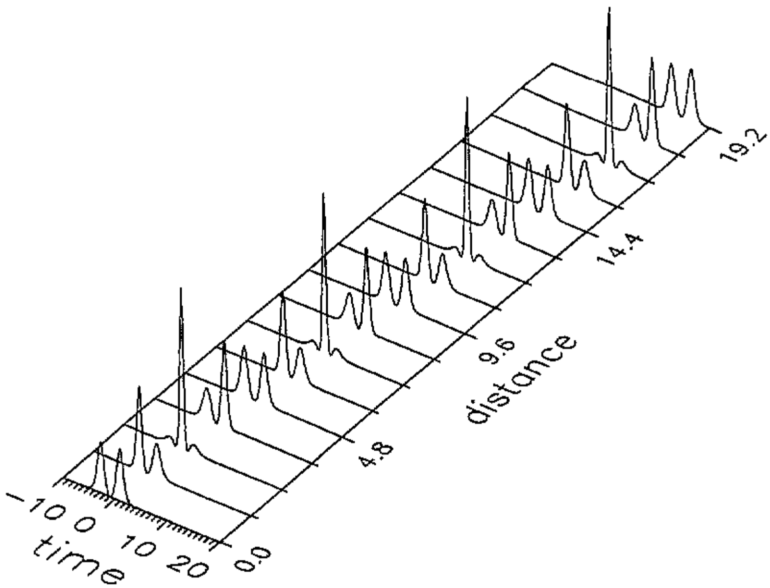

Figure 1. Intensity profile $|q|^2$ versus normalized distance Z and normalized retarded time T for the 'breather' soliton. Here $a_0 = b_0 = 1$, $\xi = 0.3$, $\eta = 1$.

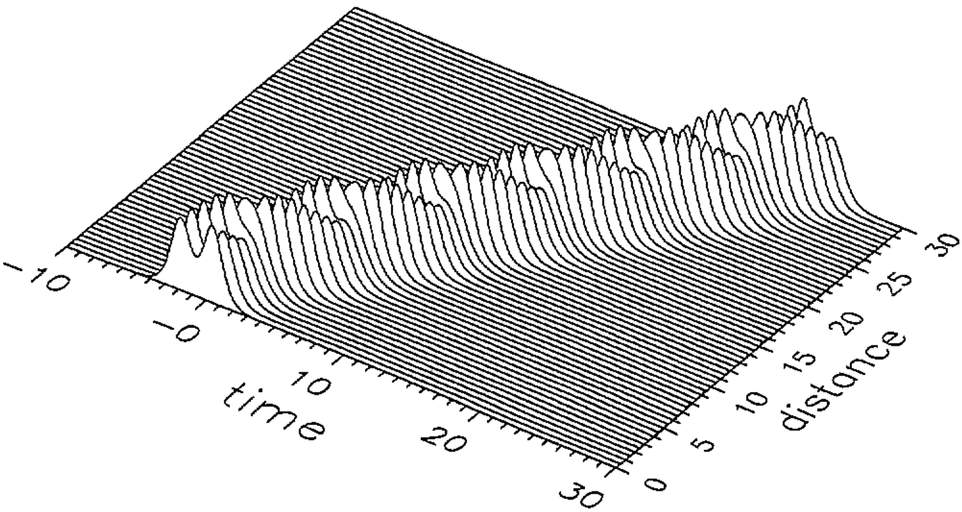

Figure 2. Intensity profile $|q|^2$ versus normalized distance Z and normalized retarded time T for a general single soliton. Here $a_0 = 1$, $b_0 = 0.2$, $\xi = 0.3$, $\eta = 1$.

References

[1] Hasegawa A and Tappert F 1973 *Appl. Phys. Lett.* **23** 142, 171

[2] Mollenauer L F, Stolen R H and Gordon J P 1980 *Phys. Rev. Lett.* **45** 1095

[3] Emplit P, Hamaide J P, Reynaud F, Froehly C and Barthelemy A 1987 *Opt. Commun.* **62** 374
Kröke1 D, Halas N J, Giuliani G and Grischkowski D 1988 *Phys. Rev. Lett.* **60** 29
Weiner A M, Heritage J P, Hawkins R J, Thurston R N, Kirschner R M, Leaird D E and Tomlinson W J 1988 *Phys. Rev. Lett* **61** 2445

[4] Zakharov V E and Shabat A B 1971 *Zh. Eksp. Teor. Fiz.* **61** 118 (Engl. Transl. 1972 *Sov.Phys.–JETP* **34** 62); 1973 *Zh. Eksp. Teor. Fiz.* **64** 1627 (Engl. Transl. 1973 *Sov.Phys.–JETP* **37** 823)
[5] Satsuma J and Yajima N 1974 *Progr. Theor. Phys. Suppl.* **55** 284
[6] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53** 249
[7] Drazin P G and Johnson R S 1989 *Solitons: An Introduction* (Cambridge: Cambridge University Press)
[8] Faddeev L D and Takhtajan L A 1987 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)
[9] Zakharov V E, Manakov S V, Novikov S P and Pitaievski L P 1984 *Theory of Solitons. The Inverse Problem Method* (New York: Plenum)
[10] Gagnon L and Winternitz P 1988 *J. Phys. A: Math. Gen.* **21** 1493; 1989 *J. Phys. A: Math. Gen.* **22** 469
Florjanczyk M and Gagnon L 1990 *Phys. Rev.* A **41** 4478
[11] Akhmediev N N, Eleonskii V M and Kulagin N E 1987 *Teor. Mat. Fiz.* **72** 183 (Engl. Transl. 1987 *Theor. Math. Phys.* **72** 809)
[12] Mihalache D and Panoiu N-C 1992 *Phys. Rev.* A **45** 6730; 1993 *J. Phys. A: Math. Gen.* **26** 2679
[13] Mihalache D, Lederer F and Baboiu D-M 1993 *Phys. Rev.* A **47** 3285
Akhmediev N N and Ankiewicz A 1993 *Phys. Rev.* A **47** 3213
Gagnon L 1993 *J. Opt. Soc. Am.* B **10** 469
[14] Kivshar Yu S and Malomed B A 1989 *Rev. Mod. Phys.* **61** 763
Elgin J N 1993 *Phys. Rev.* A **47** 4331
[15] Kaup D J 1976 *SIAM J. Appl. Math.* **31** 121; 1990 *Phys. Rev.* A **42** 5689; 1991 *Phys. Rev.* A **44** 4582
[16] Kodama Y 1985 *J. Stat. Phys.* **39** 597
Kodama Y and Hasegawa A 1987 *IEEE J. Q. Electron.* **QE-23** 510
[17] Potasek M J 1989 *J. Appl. Phys.* **65** 941
Potasek M J and Tabor M 1991 *Phys. Lett.* **154A** 449
Potasek M J 1993 *IEEE J. Quantum. Electron.* **QE-29** 281
[18] Singer F, Potasek M J, Fang J M and Teich M C 1992 *Phys. Rev.* A **46** 4192
[19] Hasegawa A 1990 *Optical Solitons in Fibers* (Berlin: Springer)
Agrawal G P 1989 *Nonlinear Fiber Optics* (Boston, MA: Academic)
[20] Mitschke F M and Mollenauer L F 1986 *Opt. Lett.* **11** 659
Gordon J P 1986 *Opt. Lett.* **11** 662
[21] Tzoar N and Jain M 1979 *Phys. Rev.* A **23** 1266
[22] Anderson D and Lisak M 1983 *Phys. Rev.* A **27** 1393
[23] Hodel W and Weber H P 1987 *Opt. Lett.* **12** 924
[24] Gouveia-Neto A S, Gomes A S L and Taylor J R 1988 *IEEE J. Quantum. Electron.* **QE-24** 332
[25] Tai K, Hasegawa A and Bekki N 1988 *Opt. Lett.* **13** 392
[26] Weiner A M, Thurston R N, Tomlinson W J, Heritage J P, Leaird D E, Kirschner E M and Hawkins R J 1989 *Opt. Lett.* **14** 868
[27] Kivshar Yu S 1993 *IEEE J. Q. Electron.* **QE-29** 250
Kivshar Yu S and Afanasjev V V 1991 *Opt. Lett.* **16** 285
Uzunov I M and Gerdjikov V S 1993 *Phys. Rev.* A **47** 1582
[28] Wai P K A, Menyuk C R, Lee Y C and Chen H H 1986 *Opt. Lett.* **11** 464
Wai P K A, Chen H H and Lee Y C 1990 *Phys. Rev.* A **41** 426
[29] Kaup D J and Newell A C 1978 *J. Math. Phys.* **19** 798
[30] Chen H H, Lee Y C and Liu C S 1979 *Phys. Scr.* **20** 490
[31] Hirota R 1973 *J. Math. Phys.* **14** 805
[32] Sasa N and Satsuma J 1991 *J. Phys. Soc. Japan* **60** 409
Malomed B A, Sasa N and Satsuma J 1991 *Chaos, Solitons and Fractals* **1** 383
[33] Mihalache D, Torner L, Moldoveanu F, Panoiu N-C and Truta N 1993 *J. Phys. A: Math. Gen.* **26** L757
[34] Zakharov V E and Manakov S V 1975 *Zh. Eksp. Teor. Fiz.* **69** 1654 (Engl. Transl. 1976 *Sov. Phys.–JETP* **42** 842)
Manakov S V 1973 *Zh. Eksp. Teor. Fiz.* **65** 505 (Engl. Transl. 1974 *Sov. Phys.–JETP* **38** 248)
[35] Kaup D J 1976 *Stud. Appl. Math.* **55** 9
[36] Khasilev V Ya 1993 *Zh. Tekh. Fiz.* **63** 98 (Engl. Transl. 1993 *J. Tech. Phys.* **38** 315)
[37] Kivshar Yu S 1989 *Physica* **40D** 11